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Uraltsev sum rule in Bakamjian–Thomas quark models

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Abstract

We show that the sum rule recently proved by Uraltsev in the heavy quark limit of QCD holds in relativistic quark models à la Bakamjian and Thomas, that were already shown to satisfy Isgur–Wise scaling and Bjorken sum rule. This new sum rule provides a *rationale* for the lower bound of the slope of the elastic IW function $\rho^2 \geqslant 3/4$ obtained within the BT formalism some years ago. Uraltsev sum rule suggests an inequality $|\tau_{3/2}(1)| > |\tau_{1/2}(1)|$. This difference is interpreted in the BT formalism as due to the Wigner rotation of the light quark spin, independently of a possible LS force. In BT models, the sum rule convergence is very fast, the $n = 0$ state giving the essential contribution in most of the phenomenological potential models. We underline that there is a serious problem, in the heavy quark limit of QCD, between theory and experiment for the decays $B \to D^*_{0,1}(\text{broad})\ell\nu$, independently of any model calculation.

1. Introduction

Recently, Uraltsev [1] has established, in the heavy quark limit of QCD, a new sum rule. The demonstration of the sum rule (SR) follows from the OPE applied to the scattering amplitude $T(\varepsilon, \mathbf{v}, \mathbf{v} - \mathbf{v}')$ in the Shifman–Voloshin limit. The function $T(\varepsilon, \mathbf{v}, \mathbf{v} - \mathbf{v}')$, where ε is the energy variable ($\varepsilon = 0$ for elastic transitions of a free quark), is the Fourier transform of the expectation value

$$\langle B^*(\mathbf{v} - \mathbf{v}')\big|T\big(J^+(0)J(x)\big)\big|B^*(0)\rangle, \tag{1}$$

where the initial state is at rest and the final state has a momentum $m_Q(\mathbf{v} - \mathbf{v}')$, $-m_Q\mathbf{v}$ being the momentum transfer carried by the intermediate states. The novelty in Uraltsev procedure is to allow a momentum for the final state in (1). Then, the function $T(\varepsilon, \mathbf{v}, \mathbf{v} - \mathbf{v}')$ can be decomposed into symmetric and antisymmetric parts $h_\pm(\varepsilon)$ in $\mathbf{v}$, $\mathbf{v}'$. The zero order moment of $h_+(\varepsilon)$ leads to Bjorken SR [2] involving ρ^2, the slope of the elastic IW function $\xi(w)$:

$$\rho^2 = \frac{1}{4} + \sum_n \big|\tau_{1/2}^{(n)}(1)\big|^2 + 2\sum_n \big|\tau_{3/2}^{(n)}(1)\big|^2 \tag{2}$$

while the zero order moment of $h_-(\varepsilon)$ leads to the new SR [1]:

$$\sum_n \big|\tau_{3/2}^{(n)}(1)\big|^2 - \sum_n \big|\tau_{1/2}^{(n)}(1)\big|^2 = \frac{1}{4}. \tag{3}$$

From (2) and (3) one gets the lower bound

$$\rho^2 \geqslant \frac{3}{4}. \tag{4}$$

The simple relations that come out immediately from (2) and (3),

$$\sum_n \big|\tau_{3/2}^{(n)}(1)\big|^2 = \frac{\rho^2}{3}, \tag{5}$$

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$$\sum_n \left|\tau_{1/2}^{(n)}(1)\right|^2 = \frac{1}{3}\left(\rho^2 - \frac{3}{4}\right) \tag{6}$$

deserve a comment. One can see that $\sum_n |\tau_{3/2}^{(n)}(1)|^2$ is proportional to ρ^2 and that $\sum_n |\tau_{1/2}^{(n)}(1)|^2$ is proportional to the *deviation* of ρ^2 from the lower bound $3/4$. Then, there is little room left for $\sum_n |\tau_{1/2}^{(n)}(1)|^2$, as it has been pointed out recently from a SR obtained for the subleading function $\xi_3(1)$ [3]:

$$\xi_3(1) = 2\left[\sum_n \Delta E_{3/2}^{(n)}\left|\tau_{3/2}^{(n)}(1)\right|^2 - \sum_n \Delta E_{1/2}^{(n)}\left|\tau_{1/2}^{(n)}(1)\right|^2\right]. \tag{7}$$

This sum rule, combined with Voloshin sum rule [4]

$$\bar{\Lambda} = 2\sum_n \Delta E_{1/2}^{(n)}\left|\tau_{1/2}^{(n)}(1)\right|^2 + 4\sum_n \Delta E_{3/2}^{(n)}\left|\tau_{3/2}^{(n)}(1)\right|^2 \tag{8}$$

yields

$$\sum_n \Delta E_{3/2}^{(n)}\left|\tau_{3/2}^{(n)}(1)\right|^2 = \frac{1}{6}[\bar{\Lambda} + \xi_3(1)], \tag{9}$$

$$\sum_n \Delta E_{1/2}^{(n)}\left|\tau_{1/2}^{(n)}(1)\right|^2 = \frac{1}{6}[\bar{\Lambda} - 2\xi_3(1)]. \tag{10}$$

Ignoring short distance QCD corrections, QCD sum rules predict, independently of all sum rule parameters [5]

$$\xi_3(1) = \frac{\bar{\Lambda}}{3} \tag{11}$$

giving

$$\frac{\sum_n \Delta E_{1/2}^{(n)}|\tau_{1/2}^{(n)}(1)|^2}{\sum_n \Delta E_{3/2}^{(n)}|\tau_{3/2}^{(n)}(1)|^2} = \frac{1}{4}. \tag{12}$$

Since the LS coupling is small, we see that we have the same trend of inequality between $\sum_n |\tau_{3/2}^{(n)}(1)|^2$ and $\sum_n |\tau_{1/2}^{(n)}(1)|^2$ as in Eqs. (5) and (6).

2. Uraltsev sum rule in Bakamjian–Thomas quark models

One of the aims of this note is to show that the SR (3) follows within quark models à la Bakamjian and Thomas. Quark models of hadrons with a fixed number of constituents, based on the Bakamjian–Thomas (BT) formalism [6,7], yield form factors that are covariant and satisfy Isgur–Wise (IW) scaling [8] in the heavy mass limit. In this class of models, the lower bound (4) was predicted some years ago [6]. Moreover, this approach satisfies the Bjorken SR that relates the slope of the IW function to the P-wave IW functions $\tau_{1/2}(w)$, $\tau_{3/2}(w)$ at zero recoil [9]. In this approach were also computed the P-wave meson wave functions and the corresponding inelastic IW functions [10], and a numerical study of ρ^2 in a wide class of models of the meson spectrum was performed (each of them characterized by an ansatz for the mass operator M, i.e., the dynamics of the system at rest) [11], together with a phenomenological study of the elastic and inelastic IW functions and the corresponding rates for $B \to D, D^*, D^{**}\ell\nu$. Moreover, the calculation of decay constants of heavy mesons within the same approach was also performed [12].

The first demonstration of Uraltsev SR within the BT quark models is rather short, relying on formulas established in Ref. [10]. Two other demonstrations will follow that will exhibit the underlying physics. The starting point is [10]:

$$\tau_j^{(n)}(1) = \int \frac{p^2\,dp}{(2\pi)^2}\varphi_j^{(n)*}(p)F_j(p), \tag{13}$$

where

$$F_{1/2}(p) = -\frac{1}{3\sqrt{3}}\left\{\varphi(p)\frac{p^2}{m+p_0}\left(3+\frac{m}{p_0}\right) + 2pp_0\frac{d\varphi}{dp}\right\},$$
$$F_{3/2}(p) = -\frac{1}{3\sqrt{3}}\left\{\varphi(p)\frac{p^2}{m+p_0}\frac{m}{p_0} + 2pp_0\frac{d\varphi}{dp}\right\} \tag{14}$$

with the radial part of the $L=1$ wave functions normalized according to

$$\frac{1}{6\pi^2}\int p^2\,dp\left[p\varphi_j^{(n)}(p)\right]^2 = 1 \tag{15}$$

and m, $p = |\mathbf{p}|$ and $p_0 = \sqrt{p^2+m^2}$ are the mass, momentum and energy of the spectator quark.

From (13), using closure in the sectors of definite $j = 1/2, 3/2$ one finds (page 325 of Ref. [10]):

$$\sum_n \left|\tau_j^{(n)}(1)\right|^2 = \frac{3}{8\pi^2}\int dp\,|F_j(p)|^2. \tag{16}$$

From (14)–(16), the expression for the difference in the left-hand side of (3) can be integrated by parts, yielding, after some algebra:

$$\sum_n \left|\tau_{3/2}^{(n)}(1)\right|^2 - \sum_n \left|\tau_{1/2}^{(n)}(1)\right|^2 = \frac{1}{8\pi^2}\int p^2\,dp\,[\varphi(p)]^2 = \frac{1}{4}, \tag{17}$$

where the last equality follows from the ground state wave function normalization [6].

Therefore, the SR (17) within the BT quark models provides a *rationale* for the lower bound $\rho^2 \geqslant 3/4$ that was found within this class of models [6]. The sum rule also establishes that the sum over the $j = 3/2$ states dominates over the one over the $j = 1/2$.

The second demonstration, that follows more closely Uraltsev proof, will illustrate quark–hadron duality. Let us first remind the proof of Bjorken SR that was given in [9]. It was shown that the *spin averaged* hadronic tensor in the BT formalism is, *in the heavy quark limit* for the active quark, identical to the free quark hadronic tensor:

$$\bar{h}_{\mu\nu}(\mathbf{v},\mathbf{v}') = \bar{h}_{\mu\nu}^{\text{free quark}}(\mathbf{v},\mathbf{v}'). \tag{18}$$

From this relation, Bjorken SR follows. In Eq. (18), the free quark tensor is

$$\bar{h}_{\mu\nu}^{\text{free quark}}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\sum_{s_1,s_1'}\left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\mu u_{s_1}(\mathbf{v})\right] \times \left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\nu u_{s_1}(\mathbf{v})\right]^* \tag{19}$$

and the hadronic tensor writes

$$\bar{h}_{\mu\nu}(\mathbf{v},\mathbf{v}') = \frac{1}{2J+1}\sum_\lambda\sum_n \langle \mathbf{P},\lambda|J_\nu|n,\mathbf{P}'\rangle \times \langle n,\mathbf{P}'|J_\mu|\mathbf{P},\lambda\rangle, \tag{20}$$

where J, λ are the spin and spin projection of the hadron of momentum $\mathbf{P}$.

In BT models, the hadronic tensor can be written [9]:

$$\bar{h}_{\mu\nu}(\mathbf{v},\mathbf{v}') = \frac{1}{2J+1}\sum_\lambda\sum_{s_{1f},s_1',s_{1i}}\left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\mu u_{s_{1i}}(\mathbf{v})\right] \times \left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\nu u_{s_{1f}}(\mathbf{v})\right]^* f^{\lambda\lambda}_{s_{1f}s_{1i}}, \tag{21}$$

where $f^{\lambda\lambda}_{s_{1f}s_{1i}}$ is the hadronic overlap:

$$f^{\lambda\lambda}_{s_{1f}s_{1i}} = \sum_{s_2}\int d^3\mathbf{p}_2\,\psi^{\lambda *}_{s_{1f}s_2}(\mathbf{P}-\mathbf{p}_2,\mathbf{p}_2) \times \psi^{\lambda}_{s_{1i}s_2}(\mathbf{P}-\mathbf{p}_2,\mathbf{p}_2) \tag{22}$$

and (18) follows from (21) and (22). The wave function $\psi^\lambda_{s_1,s_2}(\mathbf{P}-\mathbf{p}_2,\mathbf{p}_2)$ is the internal moving ground state wave function, with the active quark labelled 1 and λ being the spin projection along some axis. It is defined by deleting the momentum conserving δ-function from the total wave function. In the BT model, it is obtained from a P-depending transformation on the *rest* internal wave function.

To proceed like Uraltsev, one must generalize the hadronic tensor, allowing for different velocities and angular momentum projections. Let us consider the *polarized* hadronic tensor:

$$h^{\lambda_i\lambda_f}_{\mu\nu}(\mathbf{v}_i,\mathbf{v}_f,\mathbf{v}') = \sum_n \langle \mathbf{P}_f,\lambda_f|J_\nu|n,\mathbf{P}'\rangle \times \langle n,\mathbf{P}'|J_\mu|\mathbf{P}_i,\lambda_i\rangle. \tag{23}$$

In the BT formalism, this tensor writes, using closure and heavy mass limit [9]: [2]

$$h^{\lambda_i\lambda_f}_{\mu\nu}(\mathbf{v}_i,\mathbf{v}_f,\mathbf{v}') = \sum_{s_{1f},s_1',s_{1i}}\left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\mu u_{s_{1i}}(\mathbf{v}_i)\right] \times \left[\bar{u}_{s_1'}(\mathbf{v}')\gamma_\nu u_{s_{1f}}(\mathbf{v}_f)\right]^* f^{\lambda_f\lambda_i}_{s_{1f}s_{1i}}(\mathbf{P}_i,\mathbf{P}_f) \tag{24}$$

with the hadronic overlap

$$f^{\lambda_f\lambda_i}_{s_{1f}s_{1i}}(\mathbf{P}_i,\mathbf{P}_f) = \sum_{s_2}\int d^3\mathbf{p}_2\;\psi^{\lambda_f *}_{s_{1f}s_2}(\mathbf{P}_f-\mathbf{p}_2,\mathbf{p}_2) \times \psi^{\lambda_i}_{s_{1i}s_2}(\mathbf{P}_i-\mathbf{p}_2,\mathbf{p}_2). \tag{25}$$

In this expression $\psi^{\lambda_i}_{s_{1i}s_2}(\mathbf{P}_i-\mathbf{p}_2,\mathbf{p}_2)$ ($i \to f$ likewise) is the internal moving ground state meson wave function, and the active quark is labelled 1.

Let us choose, like Uraltsev, the vector meson B^* as initial and final state, with $\mathbf{P}_i = 0$, $\lambda_i = 0$, $\lambda_f = +1$, and the vector current with $\mu = \nu = 0$. We are thus

[2] The states $|n,\mathbf{P}'\rangle$ form a complete set of eigenfunctions at fixed $\mathbf{P}'$: $\sum_n |n,\mathbf{P}'\rangle\langle n,\mathbf{P}'| = 1$.

considering the object

$$h_{00}^{0,+1}(\mathbf{v}_i,\mathbf{v}_f,\mathbf{v}') = \sum_{s_{1f},s'_1,s_{1i}} \left[\bar{u}_{s'_1}(\mathbf{v}')\gamma_0 u_{s_{1i}}(0)\right] \times \left[\bar{u}_{s'_1}(\mathbf{v}')\gamma_0 u_{s_{1f}}(\mathbf{v}_f)\right]^* \times f^{+1,0}_{s_{1f}s_{1i}}(0,\mathbf{P}_f) \tag{26}$$

to first order in $\mathbf{v}'$, $\mathbf{v}_f$. There are, in principle, two kinds of terms contributing to this quantity:

(1) Spin-flip term coming from the active quark, i.e., from the quark current matrix element at the desired order $\bar{u}_{s'_1}(\mathbf{v}')\gamma_0 u_{s_{1f}}(\mathbf{v}_f) \sim \mathbf{v}' \times \mathbf{v}_f$ while $\bar{u}_{s'_1}(\mathbf{v}')\gamma_0 u_{s_{1i}}(0)$ cannot give a spin flip because $\mathbf{v}_i = 0$. At the desired order, one can also take the hadronic overlap at $\mathbf{P}_i = \mathbf{P}_f = 0$:

$$h_{00}^{0,+1}(0,\mathbf{v}_f,\mathbf{v}') = [\bar{u}_{-1/2}(\mathbf{v}')\gamma_0 u_{-1/2}(0)] \times [\bar{u}_{-1/2}(\mathbf{v}')\gamma_0 u_{+1/2}(\mathbf{v}_f)]^* \times f^{+1,0}_{s_{1f}s_{1i}}(0,0). \tag{27}$$

One obtains

$$h_{00}^{0,+1}(0,\mathbf{v}_f,\mathbf{v}') = \frac{1}{4\sqrt{2}}\big(\langle\downarrow|i\sigma_1\cdot(\mathbf{v}'\times\mathbf{v}_f)|\uparrow\rangle\big)^*, \tag{28}$$

where the factor $1/\sqrt{2}$ comes from the hadronic overlap, and 1 labels the active quark.

(2) Terms without spin-flip of the active quark. Then, to have a contribution to (26), one needs to appeal to a Wigner rotation of the spectator quark 2, giving a contribution $\sim \mathbf{p}_2 \times \mathbf{P}_f$. But, by integration, this term is zero, because there is no other hadron momentum than $\mathbf{P}_f$—in the hadronic overlap there is no dependence on $\mathbf{P}'$.

We are then left with expression (28), that means that we have exact duality, just like in the unpolarized, $\mathbf{P}_i = \mathbf{P}_f$ case:

$$h_{00}^{0,+1}(0,\mathbf{v}_f,\mathbf{v}') = \left[h_{00}^{0,+1}(0,\mathbf{v}_f,\mathbf{v}')\right]_{\text{free quark}}. \tag{29}$$

We need now to compute the same hadronic tensor (23) in terms of the phenomenological Isgur–Wise functions $\tau_j(w)$, within the same approximations. After a good deal of algebra, we find, using the definitions of [14], and taking into account that the states are not normalized according to the usual normalization $\langle\mathbf{v}'|\mathbf{v}\rangle = \sqrt{4v^0v'^0}\,\delta(\mathbf{v}-\mathbf{v}')$ but by $\langle\mathbf{v}'|\mathbf{v}\rangle = \delta(\mathbf{v}-\mathbf{v}')$,

$$\begin{aligned} h_{00}^{0,+1}&(\mathbf{v}_f,\mathbf{v}',\mathbf{v}_i) \\ &\cong v_f^z\frac{1}{\sqrt{2}}\big(v'^x - iv'^y\big) \\ &\quad\times\big[C\big(0^+, j=\tfrac{1}{2}\big) + C\big(1^+, j=\tfrac{1}{2}\big) \\ &\qquad + C\big(1^+, j=\tfrac{3}{2}\big) + C\big(2^+, j=\tfrac{3}{2}\big)\big], \end{aligned} \tag{30}$$

where the different contributions are (a sum over a radial quantum number is implicit)

$$\begin{aligned} &C\big(0^+, j=\tfrac{1}{2}\big) = 0, \\ &C\big(1^+, j=\tfrac{1}{2}\big) = -|\tau_{1/2}(1)|^2, \\ &C\big(1^+, j=\tfrac{3}{2}\big) = -\frac{1}{2}|\tau_{3/2}(1)|^2, \\ &C\big(1^+, j=\tfrac{3}{2}\big) = \frac{3}{2}|\tau_{3/2}(1)|^2 \end{aligned} \tag{31}$$

and the ground state does not contribute. One obtains,

$$\begin{aligned} h_{00}^{0,+1}&(\mathbf{v}_f,\mathbf{v}',\mathbf{v}_i) \\ &\cong v_f^z\,\frac{1}{\sqrt{2}}\big(v'^x - iv'^y\big) \\ &\quad\times\left[\sum_n\big|\tau_{3/2}^{(n)}(1)\big|^2 - \sum_n\big|\tau_{1/2}^{(n)}(1)\big|^2\right]. \end{aligned} \tag{32}$$

Identifying the expressions (28) and (32), Uraltsev SR follows.

Some words of caution about the general scope and limitations of Bakamjian–Thomas quark models are in order here. Both zero order moment sum rules, the ones of Bjorken [9] and Uraltsev are satisfied by this class of models. However, higher moment sum rules as Voloshin sum rule [4] are not satisfied. These higher moments sum rules seem to be specific to the gauge nature of QCD. Anyhow, one limitation of BT models is the following, as exposed in [6]. The Bakamjian–Thomas scheme was formulated to describe relativistic bound states with a fixed number of constituents, that form representations of the Poincaré group. However, when one considers matrix elements of currents with one active quark (the simplest ansatz), these matrix elements are not covariant in general, although a main result of the formalism is that they are covariant in the heavy quark limit. In the fact, one does not obtain a covariant expression for the Voloshin sum $4\sum_n \Delta E_{3/2}^{(n)}|\tau_{3/2}^{(n)}(1)|^2 + 2\sum_n \Delta E_{1/2}^{(n)}|\tau_{1/2}^{(n)}(1)|^2$, reflecting the non-covariance outside the heavy quark limit, by contrast to the Bjorken and Uraltsev ones, that are covariant.

3. The role of spectator quark Wigner rotations

Within quark models à la BT, the difference between $\tau_{3/2}^{(0)}(1)$ and $\tau_{1/2}^{(0)}(1)$ follows from formulas (13), (14) (with a suitable phase convention, $\tau_{3/2}^{(0)}$, $\tau_{1/2}^{(0)} \geqslant 0$)

$$\tau_{3/2}^{(0)}(1) - \tau_{1/2}^{(0)}(1) \cong \frac{1}{(2\pi)^2\sqrt{3}} \int p^2\, dp \left[p\varphi_{L=1}^{(0)}(p)\right]^* \frac{p}{p_0 + m}\varphi(p), \tag{33}$$

where $\varphi_{1/2}(p) \cong \varphi_{3/2}(p) = \varphi_{L=1}^{(0)}(p)$ (assuming small LS coupling) are the internal hadron wave functions at rest. We assume, as it is natural, that for the ground state $\varphi_{L=1}^{(0)}(p)$ is positive. One finds that $\tau_{3/2}^{(0)}(1)$ is larger than $\tau_{1/2}^{(0)}(1)$ even in the limit of vanishing LS coupling. The difference (33) has a simple physical interpretation, outlined in Ref. [11]: it is essentially due to the relativistic structure of the matrix elements in terms of the wave functions. More precisely, it is due to the light spectator *quark Wigner rotations*, i.e., a relativistic effect due to the center-of-mass boost, and not due to the difference coming from the spin-orbit force between the $1/2$ and $3/2$ internal wave functions at rest, which is small and has a rather moderate effect. On the contrary, the difference (33) is quite large, at least for the lowest $L = 1$ states, since for a constituent quark mass $m \cong 0.3$ GeV, the quantity $p/(p_0 + m)$ is of O(1).

Expression (33), that comes from a specific relativistic effect, is to be contrasted with the equality for any non-relativistic quark model with spin-orbit independent potential [13], also used in Ref. [14], that analizes $1/m_Q$ corrections:

$$\tau_{3/2}^{(n)}(1) = \tau_{1/2}^{(n)}(1). \tag{34}$$

Let us see how, in terms of internal wave functions *at rest*, the Wigner rotation of the *spectator quark* is responsible for the difference between $\tau_{3/2}^{(n)}$ and $\tau_{1/2}^{(n)}$ and for the non-vanishing of the r.h.s. of Uraltsev SR (17) within the BT formalism. In the previous demonstration of Uraltsev SR, the Wigner rotations were hidden in the moving internal wave functions, which themselves disappeared using completeness relations. We will now make those explicit by using the internal wave functions at rest, that gives a feeling of how the difference $|\tau_{3/2}^{(n)}|^2 - |\tau_{1/2}^{(n)}|^2$ comes out in the l.h.s. of Uraltsev SR. Consider a meson with the active heavy quark labelled 1 and the spectator quark labelled 2. In terms of internal wave functions, the current matrix element in the BT formalism writes (formula (27) of Ref. [9]):

$$\langle \mathbf{v}'|V_\mu(0)|\mathbf{v}\rangle = \sum_{s_1' s_1} \bar{u}_{s_1'}\gamma_\mu u_{s_1} \int d\mathbf{p}_2 \frac{\sqrt{(p_i\cdot v)(p_i'\cdot v')}}{p_2^0} \times \sum_{s_2' s_2} \varphi'^*_{s_1' s_2'}(\mathbf{k}_2')\left[D\left(R_2'^{-1}R_2\right)\right]_{s_2' s_2}\varphi_{s_1 s_2}(\mathbf{k}_2). \tag{35}$$

In this expression we see the basic ingredients of the model. There is a change of variables of the quark momenta, e.g., for the initial state $(\mathbf{p}_1, \mathbf{p}_2) \to (\mathbf{P}, \mathbf{k}_2)$, where $\mathbf{P}$ is the center-of mass momentum, and $\mathbf{k}_2$ the internal relative momentum, and likewise for the final state $(\mathbf{p}_1', \mathbf{p}_2') \to (\mathbf{P}', \mathbf{k}_2')$. The first term under the integral comes from the Jacobian of this change of variables. The matrix element $u_{s_1'}\gamma_\mu u_{s_1}$ expresses the fact that the quark 1 is the active heavy quark. The relation between, e.g., k_2 and p_2 is given by the boost $k_2^0 = v^0 p_2^0 - v^z p_2^z$, $k_2^z = v^0 p_2^z - v^z p_2^0$, $k_2^{x,y} = p_2^{x,y}$, v being the four-velocity of the initial state. The wave functions φ and φ' are the initial and final internal wave functions at rest, dependent only on the relative momenta and Pauli spinors. Finally, the matrix $D(R_2'^{-1}R_2)$ is the Wigner rotation acting on the spin of the spectator quark 2 due to the product of the boosts on the initial and final states. Formula (35) leads to the difference (33) and to the r.h.s. of Uraltsev SR (17). Expanding the fourth component vector current matrix element between the ground state and $L = 1$ states up to the first power of $\mathbf{v}$, $\mathbf{v}'$ gives, from (35) (formula (29) of Ref. [9]):

$$\langle n(\mathbf{v}')|V_0(0)|0(\mathbf{v})\rangle \cong \frac{1}{2}(\mathbf{v}' - \mathbf{v}) \times \left(n\right| - i\left(p_2^0\mathbf{r}_2 + \mathbf{r}_2 p_2^0\right) + \frac{i(\sigma_2\times\mathbf{p}_2)}{p_2^0 + m}\left|0\right), \tag{36}$$

where $|0(\mathbf{v})\rangle$ stands for the ground state wave function in motion and likewise $|0)$ for the internal ground state at rest in terms of Pauli spinors. The first operator $-i(p_2^0\mathbf{r}_2+\mathbf{r}_2p_2^0)$, where $\mathbf{r}_2$ is the operator $i\,\partial/\partial\mathbf{p}_2$, comes from the variation of the Jacobian factor and the variation of the argument $\mathbf{k}$ of the wave function, while the second operator $i(\sigma_2\times\mathbf{p}_2)/(p_2^0+m)$ is the Wigner rotation. Eq. (36) becomes, in the nonrelativistic limit, the matrix element of the electric dipole operator, and leads to the difference (33) through the latter spin-dependent term. To demonstrate Uraltsev SR, we are interested in the hadronic tensor

$$h_{00}^{+10}(\mathbf{v}_f,\mathbf{v}',\mathbf{v}_i)=\sum_n\big\langle B^{*(+1)}(\mathbf{v}_f)\big|V_0(0)\big|n(\mathbf{v}')\big\rangle \times\big\langle n(\mathbf{v}')\big|V_0(0)\big|B^{*(0)}(\mathbf{v}_i)\big\rangle. \tag{37}$$

The ground state does not contribute to the sum rule over intermediate states in (37), in HQET and likewise in BT quark models, that satisfy HQET. We have indeed demonstrated in Ref. [6] (formulas (26)–(29)) that BT quark models in the heavy quark limit satisfy HQET relations for all ground state form factors. More specifically, in BT quark models, as follows after some algebra from (35), the contributions of the active quark (28) cancels with the one of the spectator quark for the ground state. We are then left with the $L=1$ intermediate states for which we apply formula (36).

Defining the frame $v_i=(1,0,0,0)$, $v_f=(v_f^0,0,0,v_f^z)$, the hadronic tensor can then be written, at first order in the velocities $\mathbf{v}_f$ and $\mathbf{v}'$,

$$\begin{aligned}
&h_{00}^{+10}(\mathbf{v}_f,\mathbf{v}',\mathbf{v}_i)\\
&\quad\cong\frac{1}{4}\big(B^{*(+1)}\big|\bigg\{-v_f^z\bigg[-i\big(p_2^0z_2+z_2p_2^0\big)\\
&\qquad+\frac{i(\sigma_2\times\mathbf{p}_2)_z}{p_2^0+m}\bigg]\bigg\}^+|n)\\
&\qquad\times(n|\bigg\{v'^x\bigg[-i\big(p_2^0x_2+x_2p_2^0\big)+\frac{i(\sigma_2\times\mathbf{p}_2)_x}{p_2^0+m}\bigg]\\
&\qquad+v'^y\bigg[-i\big(p_2^0y_2+y_2p_2^0\big)+\frac{i(\sigma_2\times\mathbf{p}_2)_y}{p_2^0+m}\bigg]\bigg\}\\
&\qquad\times\big|B^{*(0)}\big),
\end{aligned}\tag{38}$$

where the $|n)$ states are $L=1$. The spin flip $B^{*(0)}\to B^{*(+1)}$ can occur because of the Wigner rotation on the spectator light quark. Using completeness $\sum_n|n)(n|=1$, two kinds of terms contribute: crossed terms between a Wigner rotation and a spin-independent operator, and products of two Wigner rotations. After some algebra, the final result reads:

$$h_{00}^{+10}(\mathbf{v}_f,\mathbf{v}',\mathbf{v}_i)\cong\frac{1}{4}v_f^z\frac{1}{\sqrt{2}}\big(v'^x-iv'^y\big). \tag{39}$$

Making explicit the states $|n)$, Eq. (38) shows that the $L=1$ states contribute to the left-hand side of Uraltsev sum rule (Eq. (32)), since the operators in brackets are $\Delta L=1$.

It may seem surprising that only a spectator quark operator appears in Eq. (38), giving the same result as the previous calculation (28), where only the active quark appeared. This is due to the fact that the right-hand side of Eq. (28) or (39) comes out from a combination of three terms: $S_1+S_2+P_2$, where S (P) means the S-wave (P-wave) contribution and 1 (2) the active (spectator) quark. It turns out that $S_1=-S_2=P_2$, showing that one gets the same r.h.s. of the SR within both formalisms. The first demonstration underlines duality, since the hadronic tensor is identical to the active quark tensor. The second demonstration underlines the physical interpretation of the SR through the Wigner rotations, since the crossed terms $\Delta L=1$, $\Delta S=1$ in (38) provide the l.h.s. of the SR, giving the difference between $j=3/2$ and $j=1/2$.

4. Phenomenological remarks

From the calculations of Ref. [11] in the BT formalism for a wide class of potentials, one can see from Table 1 that Uraltsev SR converges rapidly, as well as Bjorken's one, and are almost saturated by the $n=0$ states. [3]

The Godfrey and Isgur potential [15] is the one that describes the meson spectrum in the most complete way, from light meson spectroscopy to heavy quarkonia. The agreement of the contribution of lowest $n=0$ states with the right-hand side of the SR (17) is quite striking. Within the BT class of quark models, one gets [11] a value $\rho^2\cong 1$, not inconsistent with present experimental data on the $\xi(w)$ slope, and also, consistently, with small values for $\tau_{1/2}^{(n)}(1)$.

[3] This fast convergence of the sum rules has also been observed in QCD_2 in the $N_c\to\infty$ limit [19].

Table 1
Contribution of the lowest $L = 1$ states to the Bjorken and Uraltsev sum rules and the slope of elastic IW function in BT quark models for different potentials

Quark–antiquark potential	Godfrey, Isgur [15] $(Q\overline{Q}, Q\bar{q}, q\bar{q})$	Cea, Colangelo, Cosmai, Nardulli [16]	Isgur, Scora, Grinstein, Wise [17]
$\lvert\tau_{1/2}^{(0)}(1)\rvert^2$	0.051	0.004	0.117
$\lvert\tau_{3/2}^{(0)}(1)\rvert^2$	0.291	0.265	0.305
$\frac{1}{4}+\lvert\tau_{1/2}^{(0)}(1)\rvert^2+2\lvert\tau_{3/2}^{(0)}(1)\rvert^2$	0.882	0.790	1.068
ρ^2	1.023	0.98	1.283
$\lvert\tau_{3/2}^{(0)}(1)\rvert^2-\lvert\tau_{1/2}^{(0)}(1)\rvert^2$	0.240	0.261	0.233

Table 2
Branching ratios in BT quark models for different potentials. The experimental BR for $B \to D_2(3/2)\ell\nu$ and $B \to D_1(3/2)\ell\nu$ come from ALEPH (a), DELPHI (b) and CLEO (c) data [18], with the errors added in quadrature. The last entry corresponds to DELPHI data for the wide states

Quark–antiquark potential	Godfrey–Isgur	Cea et al.	Isgur et al.	Experimental
$B \to D\ell\nu$	2.36%	2.45%	1.94%	(2.1 ± 0.2)%
$B \to D^*\ell\nu$	6.86 %	7.02%	6.07%	(5.3 ± 0.8)%
$B \to D_2(\frac{3}{2})\ell\nu$	7.0×10^{-3}	6.5×10^{-3}	7.7×10^{-3}	(a) $(2.4 \pm 1.1) \times 10^{-3}$
				(b) $(4.4 \pm 2.4) \times 10^{-3}$
				(c) $(3.0 \pm 3.4) \times 10^{-3}$
$B \to D_1(\frac{3}{2})\ell\nu$	4.5×10^{-3}	4.2×10^{-3}	4.9×10^{-3}	(a) $(7.0 \pm 1.6) \times 10^{-3}$
				(b) $(6.7 \pm 2.1) \times 10^{-3}$
				(c) $(5.6 \pm 1.6) \times 10^{-3}$
$B \to D_1(\frac{1}{2})\ell\nu$	7×10^{-4}	4×10^{-5}	1.3×10^{-3}	$(2.3 \pm 0.7) \times 10^{-2}$
$B \to D_0(\frac{1}{2})\ell\nu$	6×10^{-4}	4×10^{-5}	1.1×10^{-3}	$[D_0(\frac{1}{2}) + D_1(\frac{1}{2})]$

It is interesting to remark that, among the three potential models quoted in Table 1, only the more complete one by Godfrey and Isgur contains a LS coupling. There are indeed in this case LS splittings ($M_{3/2}^{(n)}$ different from $M_{1/2}^{(n)}$), and the wave functions are perturbed also by this piece of the interaction, giving a different behavior for the wave functions $\varphi_{3/2}^{(n)}(p)$ and $\varphi_{1/2}^{(n)}(p)$. The other models have neglected the LS splitting, although, due to the Wigner rotations, $\tau_{3/2}^{(n)}(w)$ is, of course, different from $\tau_{1/2}^{(n)}(w)$ even for these latter potentials. However, even in the case of the Godfrey–Isgur potential, the LS force is small.

In Table 2 we compare the predictions of the BT quark models for the different semileptonic decays. While the BR for the modes $B \to D_2\ell\nu$ and $B \to D_1(3/2)\ell\nu$ have the right order of magnitude, and are consistent with experiment within 1σ, the trend of the ratio $D_1(3/2)/D_2(3/2)$ is opposite to experiment. This moderate disagreement could be explained by $1/m_Q$ corrections [20]. However, in the case of the $j = 1/2$ the disagreement is very strong. QCD in the heavy quark limit predicts, according to Uraltsev SR, that the $j = 3/2$ states are dominant over the $j = 1/2$. This general trend could be hardly reversed by the small hard QCD corrections to Uraltsev [1] and

Bjorken [20] sum rules. As to the $1/m_Q$ corrections [14], their magnitude is poorly known, since the numerical estimate of Ref. [14], although the formalism is completely general, relies on a large number of dynamical hypotheses.

Another strong experimental indication of large branching ratios of a broad resonance $D_1(1/2)$ is the non-leptonic decay $B \to D_1^0(1/2)\pi$ which is found larger than the $B \to D_J(3/2)\pi$ [21]. Factorization is reasonable in such a mode and, consequently, once again, this experimental result seems to contradict that $|\tau_{3/2}(1)| > |\tau_{1/2}(1)|$.

The serious problem for the decays $B \to D_{0,1}(1/2)\ell\nu$ goes beyond the specific BT quark models and appears to be, more generally, a problem between experiment and the heavy quark limit of QCD.

5. Conclusion

We have shown that the sum rule proved recently by Uraltsev in the heavy quark limit of QCD holds in relativistic quark models à la Bakamjian and Thomas. Its physical interpretation is the Wigner rotation of the spectator light quark spin, and not a possible LS perturbation. We have underlined that, since $|\tau_{3/2}(1)| > |\tau_{1/2}(1)|$ [22], there is a serious problem between theory and experiment for the decays $B \to D^*_{0,1}(\text{broad})\ell\nu$. This problem goes beyond the BT quark models and appears to be a general one, within the heavy quark limit of QCD.

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